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THE INCOMPRESSIBLE
LAMINAR AXISYMMETRIC WAKE

S. A. Berger

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PREFACE

The asymptotic development of the two-dimensional incompressible wake far behind a flat plate is a classical problem in fluid mechanics, but it is only recently that the correct asymptotic series solution has been given. This Memorandum treats the analogous problem for axisymmetric flow, which corresponds to the wake flow behind a very slender cylinder or "needle"; however, the solution has some relevance to the wake behind any finite axisymmetric body. The results presented here should be of interest to those working in the general area of wake flows.

SUMMARY

This Memorandum considers the asymptotic development of the axisymmetric incompressible wake far behind a long, thin cylinder. This is the axisymmetric analogue of the classical two-dimensional flat-plate problem; however, the resulting solution is applicable to a wider class of axisymmetric bodies.

In Section II an asymptotic expansion in inverse powers of the axial distance downstream of the cylinder is attempted; such an expansion is suggested by the Oseen solution to the problem. The solution is calculated to the second-order term. However, the condition that the perturbation velocity go to zero exponentially far from the axis is not satisfied by the second-order term. Hence, this term must be rejected on physical grounds.

A similar difficulty arises in the two-dimensional flat-plate problem in the third-order term. Resolution of this difficulty is accomplished by adding a logarithmic term. The following term contains an indeterminate numerical factor; this indeterminacy can be shown to be associated with the neglect of the initial velocity profile in obtaining the asymptotic far-wake solution. Indeterminate factors also appear in various other terms in the series expansion, and these can be shown to be connected with the existence of eigensolutions to the equations satisfied by individual terms of the expansion.

In Section III the axisymmetric problem is reformulated, and it is shown that as in the two-dimensional problem the cause of the breakdown of the solution given in Section II is the existence of a logarithmic term in the second-order approximation. The logarithmic term is calculated explicitly, but again, the next term is indeterminate because of neglect of the initial velocity profile. The next approximation is shown to involve a $(\ln x)^2$ term which must be included for the next term to have the correct behavior far from the axis. Indeterminate factors associated with the existence of eigensolutions also appear in succeeding terms.

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I. INTRODUCTION

At the present time the complete analytic solution for the wake behind a body, slender or blunt, is not obtainable because of the essential nonlinearity of the problem and the transition from laminar to turbulent flow at some point in the wake. In a search for a more tractable problem, one is led to a consideration of incompressible laminar wakes. However, even for this restricted problem, an analytic solution valid in the entire wake is not possible. One then looks further to see if there exist portions of such flows where an analytic treatment is possible. There do, in fact, exist two such regions for certain bodies: the first of these is the near wake immediately behind a flat plate or very slender cylinder, while the second is the wake far behind either of these bodies.

For the flat plate, no recirculating zone exists beyond the trailing edge; for the slender cylinder, whatever small recirculating zone is present may often be neglected (see Ref. 1). It is then possible to obtain a series solution for small distances from the rearmost point of the body. The flat-plate near wake was first treated by Goldstein,⁽²⁾ while the solution for the very slender cylinder, or "needle," was given recently by Viviani and Berger.⁽¹⁾

The wake far behind a flat plate has been analyzed by Tollmien,⁽³⁾ Goldstein,⁽⁴⁾ and Stewartson.⁽⁵⁾ This Memorandum discusses the far wake behind the "needle." (For reasons that will be indicated later, however, the solution obtained will, in fact, be valid for the far wake behind any finite, closed axisymmetric body.) The problem will be treated by seeking an asymptotic series solution. In Section II, we will attempt a series solution in powers of x^{-n} , n taking on successive integral values; but we will also show that the second approximation, the x^{-2} term, does not exhibit the correct behavior far from the wake axis, and hence the expansion breaks down.

In Section III, following Stewartson's approach to the two-dimensional case,⁽⁵⁾ the problem is reformulated and the solution expressed in a different form, enabling us to explain the breakdown of the direct expansion of Section II and, in addition, to indicate the form of the

correct expansion. The second approximation, which was rejected in Section II, is given explicitly; the x^{-2} term in the expansion contains an undetermined constant connected in some way with the initial profile of u , which is neglected in the asymptotic development. The nature of the indeterminacy in succeeding terms, due to the neglect of the initial profile and its relationship to the existence of eigenfunction solutions, is also indicated.

For the two-dimensional flat-plate problem, the Oseen approximation for the velocity far downstream was first given by Tollmien.⁽³⁾ Goldstein⁽⁴⁾ found the next approximation (of $O(x^{-1})$), then joined the far-wake solution to his near-wake solution⁽²⁾ by a graphical method. In this way he determined the origin on the x -axis (initially arbitrary) in the far-wake solution. Goldstein also tried calculating the next approximation (of $O(x^{-3/2})$) but had to reject his solution on physical grounds because it did not have the correct exponential decay at infinity. Stewartson⁽⁵⁾ showed that this difficulty could be resolved by adding an extra term $O(x^{-3/2} \ln x)$ into the expansion; he also found that a numerical factor occurring in the term $O(x^{-3/2})$ remained indeterminate and that this indeterminacy is probably connected with the neglect of the initial velocity profile in the asymptotic solution. Stewartson also clearly discussed the existence and significance of eigensolutions in asymptotic expansions in boundary-layer problems.

There seems very little possibility in the axisymmetric case of reasonably determining the origin for x by numerically matching the "needle" near-wake solution of Viviani and Berger⁽¹⁾ with the asymptotic far-wake solution, as the near-wake solution for the "needle" does not extend far enough downstream for such a matching to be accurate. In particular, Viviani and Berger find that the range of validity in the downstream direction of the new wake solution is of the order of the cylinder radius and that for certain ranges of the parameters, it is much less than this radius. On the other hand, for the two-dimensional near wake behind a flat plate, the range of validity of the expansion used by Goldstein is an appreciable fraction of the length of the plate.

The analyses referred to above, as well as the present one, are boundary-layer analyses; that is, the usual Reynolds number is assumed

to be large enough so that boundary-layer approximations are valid in the far-wake region. The more general problem, finding solutions of the full Navier-Stokes equations at large distances from a finite body, has been treated by Chang⁽⁶⁾ for two-dimensional flow and by Childress⁽⁷⁾ for axially symmetric flows and general three-dimensional flows. In these analyses, which are valid for a fixed value of the Reynolds number which is assumed neither large nor small, a small extraneous parameter is introduced; the construction of the asymptotic expansion is then recast as a perturbation for small values of this parameter. Due to the presence of the viscous wake, the perturbation is in general a singular one and is treated by the method of "inner and outer expansions" (or "matched asymptotic expansions"⁽⁸⁾). The results of the present analysis agree with those of Childress for the axisymmetric case, but there are significant differences in the basic approach, details of the analysis, and interpretation of certain important aspects of the problem; these are discussed at some length in Section IV. In particular, the reason for the introduction of logarithmic terms into the solution, which remains somewhat of a mystery in the artificial-parameter analysis, is explained simply by the present analysis.

There is one result of the Childress analysis which is of special significance in the formulation of the far-wake problem within the context of boundary-layer theory. The pressure gradient which appears in the boundary-layer equations would have to come from the solution outside the wake. For the "needle," as for the corresponding flat-plate problem, the pressure gradient may be taken as zero. Childress⁽⁷⁾ shows that the lowest-order contribution to the pressure gradient begins affecting the velocity distribution u at the x^{-2} term, but without knowing more about the initial profile of u than the total drag, this term is indeterminate. Hence, up to but exclusive of the term of order x^{-2} , which includes all the determinate terms in the expansion, the asymptotic expansion for u presented in this Memorandum is valid not only for the "needle," but for any finite, closed, axisymmetric body.

II. BASIC EQUATIONS AND ASYMPTOTIC SOLUTION

We introduce a cylindrical coordinate system (x_1, r_1) , where x_1 is the axial distance measured from the base of the cylinder and r_1 is the radial distance measured from the axis of symmetry. The corresponding velocities will be denoted by u_1 and v_1 , respectively.

In this coordinate system, the continuity and momentum equations without pressure gradient are

$$\frac{\partial}{\partial x_1} (r_1 u_1) + \frac{\partial}{\partial r_1} (r_1 v_1) = 0 \quad (1)$$

$$u_1 \frac{\partial u_1}{\partial x_1} + v_1 \frac{\partial u_1}{\partial r_1} = \frac{v}{r_1} \frac{\partial}{\partial r_1} \left(r_1 \frac{\partial u_1}{\partial r_1} \right) \quad (2)$$

These are to be solved subject to the boundary conditions

$$v_1 = 0 \quad \frac{\partial u_1}{\partial r_1} = 0 \quad \text{at} \quad r_1 = 0 \quad (3)$$

$$u_1 \rightarrow U \quad \text{as} \quad r_1 \rightarrow \infty \quad (4)$$

and in the limit of $x_1 \rightarrow \infty$.

Childress,⁽⁷⁾ in his asymptotic analysis of the Navier-Stokes equations for the flow at large distances from a finite axisymmetric body, has shown that the pressure in the inner viscous wake has the following asymptotic form:

$$\frac{p - P}{\rho U^2} = - \frac{a}{4\pi} \frac{1}{\bar{x}} \epsilon^2 + O(\epsilon^3 \ln \epsilon)$$

where $\bar{x} = \epsilon x/L$, P is the uniform free-stream pressure, L is a characteristic length, a is a constant related to the dimensionless drag, and ϵ , an "artificial parameter," is the ratio of the characteristic

length to the length of an extraneous standard of measurement. Physically, the leading term in this expansion is the pressure induced by a potential source, representing the effect of the displacement thickness on the outer flow. We see that far from any finite axisymmetric body the pressure, up to but not including the term of order x^{-2} , is constant; consequently, to that order the formulation of the far-wake problem as given above for zero pressure gradient applies not only to the "needle" but to finite bodies as well.

Equation (1) is satisfied by introducing a stream function ψ_1 defined by

$$r_1 u_1 = \frac{\partial \psi_1}{\partial r_1} \quad (5a)$$

$$r_1 v_1 = - \frac{\partial \psi_1}{\partial x_1} \quad (5b)$$

It is now convenient to introduce a dimensionless stream function $f(\zeta, x_1)$ such that

$$\psi_1 = v x_1 f(\zeta, x_1) \quad (6)$$

where

$$\zeta = \frac{U r_1^2}{4 v x_1} \quad (7)$$

Then the nondimensional velocity components u and v can be written:

$$u = \frac{u_1}{U} = \frac{1}{2} f_\zeta \quad (8)$$

$$v = \frac{v_1}{U} = - \frac{v}{U r_1} (f - \zeta f_\zeta + x_1 f_{x_1}) \quad (9)$$

(The subscripts are derivatives.)

In terms of these new independent and dependent variables, Eq. (2) becomes

$$\zeta f_{\zeta\zeta\zeta} + f_{\zeta\zeta} + \frac{1}{2} f f_{\zeta\zeta} + \frac{1}{2} x_1 f_{x_1} f_{\zeta\zeta} - \frac{1}{2} x_1 f_{x_1} \zeta f_\zeta = 0 \quad (10)$$

Now x_1 is nondimensionalized by means of the momentum thickness θ_1 , defined by

$$\pi \theta_1^2 U^2 = 2\pi \int_0^\infty (U - u_1) u_1 r_1 dr_1 \quad (11)$$

or

$$\theta_1^2 = \frac{4\nu x_1}{U} \int_0^\infty (1 - u) u d\zeta \quad (12)$$

Here θ_1 is a constant, related to the drag of the body, D_1 , by

$$D_1 = \pi \rho_1 U^2 \theta_1^2 \quad (13)$$

We now set

$$x = \frac{x_1}{\theta_1} \quad (14)$$

The differential equation Eq. (10), however, is unaffected by this change, since it is homogeneous in x_1 .

For the asymptotic solution, we assume that $f(\zeta, x)$ has an expansion for large values of x of the form

$$f(\zeta, x) = \sum_{n=0}^{\infty} \frac{f_n(\zeta)}{x^n} \quad (15)$$

Substituting into Eq. (10) yields

$$\zeta \sum_{n=0}^{\infty} \frac{f_n'''}{x^n} + \sum_{n=0}^{\infty} \frac{f_n''}{x^n} + \frac{1}{2} \sum_{n=0}^{\infty} \frac{f_n}{x^n} \sum_{n=0}^{\infty} \frac{f_n''}{x^n} - \frac{1}{2} \sum_{n=0}^{\infty} \frac{nf_n}{x^n} \sum_{n=0}^{\infty} \frac{f_n''}{x^n} + \frac{1}{2} \sum_{n=0}^{\infty} \frac{nf_n'}{x^n} \sum_{n=0}^{\infty} \frac{f_n'}{x^n} = 0 \quad (16)$$

Equating the coefficients of powers of x^n to zero yields the following set of ordinary differential equations:

$$\zeta f_0''' + f_0'' + \frac{1}{2} f_0 f_0'' = 0 \quad (17)$$

$$\zeta f_1''' + f_1'' + \frac{1}{2} f_0 f_1'' + \frac{1}{2} f_0' f_1' = 0 \quad (18)$$

$$\zeta f_2''' + f_2'' + \frac{1}{2} f_0 f_2'' - \frac{1}{2} f_0' f_2' + f_0' f_2' + \frac{1}{2} f_1'^2 = 0 \quad (19)$$

$$\zeta f_3''' + f_3'' + \frac{1}{2} f_0 f_3'' + \frac{3}{2} f_0' f_3' - \frac{1}{2} f_2 f_1'' - f_0' f_3 + \frac{3}{2} f_1' f_2' = 0 \quad (20)$$

Since

$$\psi_1 = v x_1 \sum_{n=0}^{\infty} \frac{f_n(\zeta)}{(x_1/\theta_1)^n}$$

we note that the boundary conditions, Eq. (3), require

$$f_n(0) = 0 \quad (21)$$

$$\sqrt{\zeta} f_n''(\zeta) \rightarrow 0 \quad \text{as} \quad \zeta \rightarrow 0 \quad (22)$$

Boundary-condition Eq. (4) implies that

$$f_\zeta = \sum_{n=0}^{\infty} \frac{f_n'(\zeta)}{x^n} \rightarrow 2 \quad \text{as} \quad \zeta \rightarrow \infty \quad (23)$$

The condition

$$u_1 \rightarrow U \quad \text{as} \quad x_1 \rightarrow \infty \quad \text{independent of } \zeta \quad (24)$$

must also be satisfied, and this indicates that

$$f_0'(\zeta) \rightarrow 2 \quad \text{as} \quad \zeta \rightarrow \infty \quad (25)$$

Equation (23) then reduces to the condition

$$f_n'(\zeta) \rightarrow 0 \quad \text{as} \quad \zeta \rightarrow \infty \quad n \geq 1 \quad (25a)$$

SOLUTION FOR $f_0(\zeta)$

The differential equation is Eq. (17):

$$\zeta f_0''' + f_0'' + \frac{1}{2} f_0 f_0'' = 0 \quad (26)$$

with boundary conditions

$$\left. \begin{aligned} f_0(0) &= 0 \\ \sqrt{\zeta} f_0''(\zeta) &\rightarrow 0 \quad \text{as} \quad \zeta \rightarrow 0 \\ f_0'(\zeta) &\rightarrow 2 \quad \text{as} \quad \zeta \rightarrow \infty \end{aligned} \right\} \quad (27)$$

A solution satisfying the differential equation and all the boundary conditions is obvious by inspection and is

$$f_0(\zeta) = 2\zeta \quad (28)$$

SOLUTION FOR $f_1(\zeta)$

Substituting Eq. (28) into Eq. (18) for $f_1(\zeta)$, we obtain

$$\zeta f_1''' + (\zeta + 1)f_1'' + f_1' = 0 \quad (29)$$

the solution of which must satisfy the boundary conditions

$$\left. \begin{aligned} f_1(0) &= 0 \\ \sqrt{\zeta} f_1''(\zeta) &\rightarrow 0 \quad \text{as} \quad \zeta \rightarrow 0 \\ f_1'(\zeta) &\rightarrow 0 \quad \text{as} \quad \zeta \rightarrow \infty \end{aligned} \right\} \quad (30)$$

If we set

$$f_1'(\zeta) = e^{-\zeta} g_1(\zeta)$$

then $g_1(\zeta)$ satisfies the equation

$$\zeta g_1'' - (\zeta - 1)g_1' = 0$$

whose solution is

$$g_1(\zeta) = c \int \frac{e^\zeta}{\zeta} d\zeta + c_1$$

where c and c_1 are arbitrary constants.

Therefore,

$$f_1'(\zeta) = ce^{-\zeta} \int \frac{e^\zeta}{\zeta} d\zeta + c_1 e^{-\zeta}$$

It is easily seen that the second boundary condition, Eq. (30), cannot be satisfied unless $c = 0$; hence

$$f_1'(\zeta) = c_1 e^{-\zeta} \quad (31)$$

This automatically satisfies the third boundary condition. Integrating and applying the remaining boundary condition yields

$$f_1(\zeta) = c_1(1 - e^{-\zeta}) \quad (32)$$

The constant c_1 is directly related to the drag of the body through θ_1 . In particular, since θ_1 is independent of x , we can write Eq. (12) in the limit $x \rightarrow \infty$,

$$\theta_1^2 = \lim_{x_1 \rightarrow \infty} \frac{4\nu x_1}{U} \int_0^\infty \frac{-f_1'(\zeta)}{2x} d\zeta = \frac{-4\nu\theta_1}{U} \int_0^\infty \frac{c_1}{2} e^{-\zeta} d\zeta$$

and obtain

$$c_1 = -\frac{U\theta_1}{2v} \quad (33)$$

SOLUTION FOR $f_2(\zeta)$

Using Eqs. (28) and (31), we can write Eq. (19) as

$$\zeta f_2''' + (\zeta + 1)f_2'' + 2f_2' + \frac{1}{2} c_1^2 e^{-2\zeta} = 0 \quad (34)$$

subject to the boundary conditions

$$\left. \begin{aligned} f_2(0) &= 0 \\ \sqrt{\zeta} f_2''(\zeta) &\rightarrow 0 \quad \text{as} \quad \zeta \rightarrow 0 \\ f_2'(\zeta) &\rightarrow 0 \quad \text{as} \quad \zeta \rightarrow \infty \end{aligned} \right\} \quad (35)$$

Let

$$f_2'(\zeta) = (\zeta - 1)e^{-\zeta} g_2(\zeta) \quad (36)$$

Then g_2 satisfies

$$\zeta(\zeta - 1)g_2'' + [2\zeta - (\zeta - 1)^2]g_2' = -\frac{1}{2} c_1^2 e^{-\zeta}$$

This is easily integrated and yields

$$g_2'(\zeta) = \frac{c_1^2}{4} \frac{e^{\zeta}}{\zeta(\zeta - 1)^2} \left[(\zeta - \frac{1}{2})e^{-2\zeta} + c_2' \right]$$

If we integrate this result and substitute in Eq. (36), we obtain

$$f_2'(\zeta) = -\frac{c_1^2}{8} \left\{ (\zeta - 1)e^{-\zeta} \int_A^\zeta \frac{e^{-t}}{t} dt + e^{-2\zeta} \right\} \\ + c_2 \left\{ (\zeta - 1)e^{-\zeta} \int_A^\zeta \frac{e^t}{t} dt - 1 \right\}$$

where A and c_2 are arbitrary constants.

As $\zeta \rightarrow 0$

$$\int_A^\zeta \frac{e^{-t}}{t} dt \sim \ln \zeta$$

$$\int_A^\zeta \frac{e^t}{t} dt \sim \ln \zeta$$

which means that $f_2'(\zeta) \rightarrow \infty$ as $\ln \zeta$ when $\zeta \rightarrow 0$. but this would indicate that u is infinite along the axis, $r_1 = 0$. To avoid this, we must choose

$$c_2 = \frac{c_1^2}{8} \quad (37)$$

The boundary condition $f_2'(\zeta) \rightarrow 0$ as $\zeta \rightarrow \infty$ is satisfied, since

$$\zeta e^{-\zeta} \int_A^\zeta \frac{e^t}{t} dt \rightarrow 1 \quad \text{as} \quad \zeta \rightarrow \infty$$

One can also verify that $f_2'' \rightarrow \text{constant}$ as $\zeta \rightarrow 0$, so that

$$\sqrt{\zeta} f_2''(\zeta) \rightarrow 0 \quad \text{as} \quad \zeta \rightarrow 0$$

as required by the middle boundary condition of Eq. (35). This completes the determination of $f_2'(\zeta)$, which may be written as

$$f_2'(\zeta) = + \frac{c_1^2}{8} \left\{ (\zeta - 1)e^{-\zeta} \left[\int_0^\zeta \frac{e^t - e^{-t}}{t} dt + 2c - 2 \right] - e^{-2\zeta} - 1 \right\} \quad (38)$$

(The boundary condition $f_2(0) = 0$ is obviously superfluous as far as the determination of f_2' is concerned.) Note that after satisfying all the boundary conditions, an arbitrary constant c still remains in the equation for $f_2'(\zeta)$.

To second order then, the velocity on the x_1 axis is given by

$$u_0(x) = 1 + \frac{c_1}{2x} - c \frac{c_1^2}{8x^2} + \dots$$

where c is an undetermined constant.

III. MODIFICATION OF EXPANSION

The occurrence of an arbitrary constant in the expression for $f_2'(\zeta)$, and consequently for $u_0(x)$, is not in itself surprising. In fact, since the initial conditions, ignored in calculating the asymptotic expansion, influence the far-wake solution, we must anticipate that at some stage of the expansion indeterminacy will occur. Of greater concern is the fact that the second approximation, $f_2'(\zeta)$, does not approach zero exponentially as $\zeta \rightarrow \infty$, as we would expect, since

$$\lim_{\zeta \rightarrow \infty} \left[(\zeta - 1)e^{-\zeta} \int_A^{\zeta} \frac{e^t - e^{-t}}{t} dt \right] = \lim_{\zeta \rightarrow \infty} \left[(\zeta - 1)e^{-\zeta} \right. \\ \left. \times \left(\frac{e^{\zeta}}{\zeta} + \frac{e^{\zeta}}{\zeta^2} + \dots \right) \right] = \lim_{\zeta \rightarrow \infty} \left[1 - \frac{1}{\zeta^2} + o\left(\frac{1}{\zeta^3}\right) \right]$$

We note that $f_2'(\zeta)$ goes to zero as ζ^{-2} as $\zeta \rightarrow \infty$. A similar phenomenon was found by Goldstein⁽⁴⁾ in the two-dimensional case in the third approximation (his solution proceeded in terms of $x^{-n/2}$, n taking on all integral values). Even if this approximation were not rejected on physical grounds, it is actually impossible, as shown by Goldstein, to satisfy all the boundary conditions in the next approximation, so the solution is not even possible mathematically. It is very likely we would encounter the same difficulty if we attempted to solve for the next term in our expansion, $f_3'(\zeta)$.

Stewartson⁽⁵⁾ resolved the difficulty for the two-dimensional flat-plate case; he showed that the asymptotic expansion contained an extra term $O(x^{-3/2} \ln x)$ in addition to the $x^{-n/2}$ terms ($n = 1, 2, 3$) contained in Goldstein's original solution and also that a numerical factor occurring in the term $O(x^{-3/2})$ was indeterminate. Stewartson treated a number of boundary-layer problems involving asymptotic expansions for large x and demonstrated that these phenomena (the occurrence of logarithmic terms and indeterminacy in the following algebraic terms) are general features of all of them.

Briefly, Stewartson's demonstration proceeds as follows: In each of the problems considered at the n^{th} stage of the expansion procedure, the required function satisfies a third-order ordinary linear differential equation with, say, η as an independent variable and n as a parameter; two boundary conditions are given at $\eta = 0$ and one at $\eta = \infty$. For each of an infinite countable set of n 's, not necessarily integers, there is a complementary solution satisfying all the boundary conditions at $\eta = 0$ and at $\eta = \infty$. (These are called "eigensolutions." (6,8)) It follows that the factor multiplying this function cannot be determined; and Stewartson concludes that it probably depends on the neglected upstream boundary condition, the initial velocity profile. When a member of this set is an integer, then no particular integral of the differential equation is exponentially small for large η unless an additional term, consisting of the complementary solution multiplied by $\ln x$ or $\ln \ln x$, depending on the problem, is added. The factor in this term is determined by requiring that the particular integral be exponentially small for large η ; the particular integral so defined is, however, not unique.

In the following, we shall demonstrate the particular form which the ideas expressed above take in the present problem. Using the same general approach as Stewartson employed for the flat-plate problem, we shall indicate why the expansion of the previous section broke down and proceed to determine the correct expansion. In this process, we shall explicitly determine the set of n for which eigenfunctions exist and the nature of the eigenfunctions themselves. We shall see that there are significant differences between the two-dimensional and axisymmetric cases.

Consider Eqs. (1) and (2) again with the boundary conditions

$$\left. \begin{array}{lll} u_1 \rightarrow U & \text{as} & r_1 \rightarrow \infty \quad x_1 > 0 \\ v_1 = \frac{\partial u_1}{\partial r_1} = 0 & \text{at} & r_1 = 0 \quad x_1 > 0 \end{array} \right\} \quad (39)$$

and the initial condition

$$u_1 = Uf(r_1) \quad \text{at} \quad x_1 = 0 \quad r_1 > 0 \quad (40)$$

This initial condition is introduced here only for convenience in carrying out the later discussion, which will lead to the correct form of expansion. This initial condition should not appear in the actual formulation of the asymptotic problem. (It does, however, play a role in determining certain constants in the asymptotic solution.) In general, an axisymmetric body of nonzero thickness will have a separated region immediately behind the base, and the flow field is too complex to allow the computation of $f(r_1)$. However, for the purposes of our later discussion it is not necessary that the initial profile be given at the base of the body; it is only necessary that some initial profile be specified, and we arbitrarily place the origin of the x_1 axis at the place where the profile is given. Thus, for example, we can think of $f(r_1)$ as being given at some location downstream of the separated region if one exists. In any case, whether or not $f(r_1)$ can readily be determined is immaterial to our later arguments. However, we do assume that $f(r_1)$ is some sort of boundary-layer solution. As a corollary of this, it follows that $f(r_1) \rightarrow 1$ exponentially as r_1 approaches infinity.

We nondimensionalize by setting

$$u = \frac{u_1}{U} \quad v = \left(\frac{U\theta_1}{v} \right)^{\frac{1}{2}} \frac{v_1}{U} \quad x = \frac{x_1}{\theta_1} \quad r = \left(\frac{U\theta_1}{v} \right)^{\frac{1}{2}} \frac{r_1}{\theta_1} \quad (41)$$

Equations (1) and (2) now become

$$\frac{\partial}{\partial x} (ru) + \frac{\partial}{\partial r} (rv) = 0 \quad (42)$$

$$u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial r} = \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) \quad (43)$$

A first approximation to u when x is very large can be obtained from the Oseen approximation and the fact that the momentum thickness is constant (since the pressure is taken to be constant). The result is

$$u = 1 - \frac{A}{x} e^{-r^2/4x} \equiv u_1 \quad (44)$$

say, where A is given in terms of the nondimensional drag D by

$$D = \frac{D_1}{vU\theta_1} = 4\pi\rho A \quad (45)$$

or

$$A = \frac{1}{2} \int_0^\infty f(1-f)r \, dr \quad (46)$$

To improve this approximation we substitute Eq. (44) into Eqs. (42) and (43), thus starting an iteration. To demonstrate the nature of this process at any stage, we set

$$\left. \begin{aligned} u &= 1 + \bar{u} \\ v &= \bar{v} \end{aligned} \right\} \quad (47)$$

Substituting into Eq. (43), we rewrite it as

$$\frac{\partial \bar{u}}{\partial x} - \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \bar{u}}{\partial r} \right) = -\bar{u} \frac{\partial \bar{u}}{\partial x} - \bar{v} \frac{\partial \bar{u}}{\partial r}$$

The solution in the n^{th} iteration is then obtained from the equation

$$\frac{\partial \bar{u}^{(n)}}{\partial x} - \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \bar{u}^{(n)}}{\partial r} \right) = -\bar{u}^{(n-1)} \frac{\partial \bar{u}^{(n-1)}}{\partial x} - \bar{v}^{(n-1)} \frac{\partial \bar{u}^{(n-1)}}{\partial r} \quad (48)$$

(This iterative method of solution can be shown to be equivalent to the asymptotic series-expansion approach discussed in Section II.)

For $n = 1$, taking $\bar{u}^{(0)} = 0$, Eq. (48) reduces to

$$\frac{\partial \bar{u}^{(1)}}{\partial x} - \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \bar{u}^{(1)}}{\partial r} \right) = 0 \quad (49)$$

the solution of which is

$$\bar{u}^{(1)} = -\frac{A}{x} e^{-r^2/4x} \quad (50)$$

(already given in Eq. (44)).

When $n = 2$, Eq. (48) reads

$$\frac{\partial \bar{u}^{(2)}}{\partial x} - \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \bar{u}^{(2)}}{\partial r} \right) = -\bar{u}^{(1)} \frac{\partial \bar{u}^{(1)}}{\partial x} - \bar{v}^{(1)} \frac{\partial \bar{u}^{(1)}}{\partial r} \quad (51)$$

By substituting Eq. (50) into Eq. (42), we find that

$$\bar{v}^{(1)} = -\frac{Ar}{2x^2} e^{-r^2/4x} \quad (52)$$

Substitution of Eqs. (50) and (52) into Eq. (51) now yields as the governing equation for the second approximation

$$\frac{\partial \bar{u}^{(2)}}{\partial x} - \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \bar{u}^{(2)}}{\partial r} \right) = \frac{A^2}{x^3} e^{-r^2/2x} \quad (53)$$

Rather than determine the solution to Eq. (53) at this time, we want to consider the general problem at any stage of the iteration. In each iteration, we are treating the equation

$$\frac{\partial u}{\partial x} - \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial u}{\partial r} \right) = F(x, r) \quad (54)$$

where

$$F(x,r) = (1 - u) \frac{\partial u}{\partial x} - v \frac{\partial u}{\partial r} \quad (55)$$

the value of F being calculated using the u and v determined from the solution of Eq. (54) in the previous iteration.

Equation (54) is of the form of an inhomogeneous heat equation, and the determination of its solution is a standard problem in heat conduction. Assuming $f(r)$ and $F(x,r)$ to be even functions of r , the solution of Eq. (54) satisfying the boundary conditions, Eqs. (39) and (40), is given by

$$\begin{aligned} u = & 1 - \int_0^\infty \frac{1 - f(r')}{2x} e^{-\frac{(r^2 + r'^2)}{4x}} I_0\left(\frac{rr'}{2x}\right) r' dr' \\ & + \int_0^x dx' \int_0^\infty dr' \frac{r' F(x', r')}{2(x - x')} e^{-\frac{(r^2 + r'^2)}{4(x - x')}} I_0\left(\frac{rr'}{2(x - x')}\right) \end{aligned} \quad (56)$$

where I_0 is the modified Bessel function of the first kind and zeroth order.

For simplicity and without sacrificing any of the significant details, we shall limit ourselves to determining u on the axis, $r = 0$. Setting $r = 0$ in Eq. (56) yields

$$\begin{aligned} u|_{r=0} = & 1 - \int_0^\infty \frac{1 - f(r')}{2x} e^{-\frac{r'^2}{4x}} r' dr' \\ & + \int_0^x dx' \int_0^\infty dr' \frac{r' F(x', r')}{2(x - x')} e^{-\frac{r'^2}{4(x - x')}} \end{aligned} \quad (57)$$

With the general solution for any iteration now given in terms of quadratures, we return to the determination of $\bar{u}^{(2)}$. From Eq. (53) we note that $F(x,r)$ can be written

$$F(x,r) = \frac{A^2}{x^3} e^{-r^2/2x} = g(x)F_2(\eta) \quad (58)$$

where

$$\left. \begin{aligned} g(x) &= \frac{A^2}{x^3} \\ F_2(\eta) &= e^{-\eta^2} \\ \eta &= \frac{r}{\sqrt{2x}} \end{aligned} \right\} \quad (59)$$

Introducing Eq. (58) and the new variable η into the second integral of Eq. (57), we obtain

$$\begin{aligned} u|_{r=0} &= 1 - \int_0^\infty \frac{1 - f(r')}{2x} e^{-\frac{r'^2}{4x}} r' dr' \\ &+ \int_0^x \frac{g(x')x' dx'}{(x - x')} \int_0^\infty \eta' F_2(\eta') e^{-\frac{x'\eta'^2}{2(x-x')}} d\eta' \end{aligned} \quad (60)$$

Now we write

$$L'(x) = xg(x) = \frac{A^2}{x^2} \quad (61)$$

The contribution from F is then

$$\int_0^x \frac{L'(x')}{(x-x')} dx' \int_0^\infty \eta' F_2(\eta') e^{-\frac{x'\eta'^2}{2(x-x')}} d\eta' \quad (62)$$

Integrating by parts, this becomes

$$-\frac{L(0)}{x} \int_0^\infty \eta' F_2(\eta') d\eta' - \int_0^x \frac{L(x')}{(x-x')^2} dx' \int_0^\infty d\eta' \left[1 - \frac{x\eta'^2}{2(x-x')} \right] \\ \times \eta' F_2(\eta') e^{-\frac{x'\eta'^2}{2(x-x')}} \quad (63)$$

(We cannot conclude that $L(0)$ is infinite from Eq. (61), since the iterative solution based on Eq. (54) is valid only for large x .) For large x the leading terms in Eq. (63) are

$$-\frac{L(0)}{x} \int_0^\infty \eta' F_2(\eta') d\eta' + \frac{A^2}{x^2} \ln x \int_0^\infty \left(1 - \frac{\eta'^2}{2} \right) \eta' F_2(\eta') d\eta' \quad (64)$$

Following the method of Stewartson, two questions can now be raised. First, does the initial profile affect u only through the constant A , given in terms of $f(r)$ by Eq. (46)? Second, is it possible to find a second-order approximation which has the correct asymptotic behavior as $r \rightarrow \infty$, thus eliminating the difficulty encountered in Section II; and if so, what is the correct second approximation?

The first question may now be answered on the basis of expression (64). The first term represents a modification to the term $O(1/x)$ of the expansion for large x ; this occurs because the contribution from $f(r)$ in Eq. (60) to the $O(1/x)$ term is

$$\frac{1}{2x} \int_0^\infty (1 - f(r')) r' dr' \quad (65)$$

which differs from Eq. (46). There will be a contribution to expression (65) at each stage until the coefficient of $1/x$ is Eq. (46). In Eq. (57) the general term of the contribution from $f(r)$ to $u(x,0)$ is

$$\frac{(-1)^n}{2x(4x)^n n!} \int_0^\infty (1 - f(r')) r'^{2n+1} dr' \quad (66)$$

and in a similar way this will be modified at each stage of the iteration. Thus, we can expect terms in the expansion of order x^{-n-1} to have an indeterminate numerical factor depending on $f(r)$. The term with $n = 0$ is completely determined because of the momentum integral, but this is the only such integral known.

Expression (64) enables us to answer the second question also. The second term of expression (64) shows that the asymptotic expansion contains a term of the form $\ln x/x^2$; it was the absence of this term in the expansion in x^{-n} , considered in Section II, which caused the difficulties encountered in the second-order approximation.

Let us now return to Eq. (53) and assume that

$$\bar{u}^{(2)} = \frac{\ln x}{x^2} G(\eta) + \frac{1}{x^2} H(\eta) \quad (67)$$

Substitution in Eq. (53) yields the equations

$$\eta G'' + (\eta^2 + 1)G' + 4\eta G = 0 \quad (68)$$

$$\eta H'' + (\eta^2 + 1)H' + 4\eta H = 2\eta G - 2A^2 \eta e^{-\eta^2} \quad (69)$$

The boundary conditions are

$$\left. \begin{array}{l} G'(0) = H'(0) = 0 \\ \text{and} \\ G, H \rightarrow 0 \text{ exponentially as } \eta \rightarrow \infty \end{array} \right\} \quad (70)$$

The solution of Eq. (68) satisfying the boundary conditions, Eq. (70), is

$$G(\eta) = c \left(\frac{\eta^2}{2} - 1 \right) e^{-\eta^2/2} \quad (71)$$

Substituting this result into Eq. (69) and changing to the new independent variable $\bar{\zeta}$ defined by

$$\bar{\zeta} = \frac{\eta^2}{2} \quad (72)$$

(this $\bar{\zeta}$ is the nondimensional form of the ζ used in Section II), we obtain

$$\bar{\zeta} H'' + (\bar{\zeta} + 1) H' + 2H = c(\bar{\zeta} - 1)e^{-\bar{\zeta}} - A^2 e^{-2\bar{\zeta}} \quad (73)$$

where primes now denote differentiation with respect to $\bar{\zeta}$. This can be solved by setting

$$H(\bar{\zeta}) = (\bar{\zeta} - 1)e^{-\bar{\zeta}} K(\bar{\zeta}) \quad (74)$$

Here, $K(\bar{\zeta})$ satisfies the equation

$$K'' + \frac{[2\bar{\zeta} - (\bar{\zeta} - 1)^2]}{\bar{\zeta}(\bar{\zeta} - 1)} K' = \frac{c(\bar{\zeta} - 1) - A^2 e^{-\bar{\zeta}}}{\bar{\zeta}(\bar{\zeta} - 1)}$$

A first integral of this equation is

$$K' e^{-\bar{\zeta}} \bar{\zeta}(\bar{\zeta} - 1)^2 = \int_0^{\bar{\zeta}} e^{-\bar{\zeta}} (\bar{\zeta} - 1) \left[C(\bar{\zeta} - 1) - A^2 e^{-\bar{\zeta}} \right] d\bar{\zeta} \quad (75)$$

If $H(\eta)$ is to tend exponentially to zero as $\eta \rightarrow \infty$ ($\bar{\zeta} \rightarrow \infty$), then we must require

$$\int_0^{\infty} e^{-\bar{\zeta}} (\bar{\zeta} - 1) \left[C(\bar{\zeta} - 1) - A^2 e^{-\bar{\zeta}} \right] d\bar{\zeta} = 0 \quad (76)$$

This condition determines C to be

$$C = -\frac{1}{4} A^2 \quad (77)$$

Integrating Eq. (75) once more and substituting into Eq. (74), we obtain the $H(\bar{\zeta})$ satisfying the differential equation, Eq. (69), and boundary conditions, Eq. (70):

$$\begin{aligned} H(\bar{\zeta}) = & -\frac{A^2}{4} \left[(\bar{\zeta} - 1) e^{-\bar{\zeta}} \int_{\alpha}^{\bar{\zeta}} \frac{e^{-t}}{t} dt + e^{-2\bar{\zeta}} \right] \\ & + \frac{A^2}{4} \left[(\bar{\zeta} - 1) e^{-\bar{\zeta}} \ln \bar{\zeta} - 2e^{-\bar{\zeta}} \right] \end{aligned} \quad (78)$$

where α is an undetermined constant.

Thus, G is determined uniquely but H is not; this is not surprising, since, as we saw earlier, H probably depends on $f(r)$.

The solution for u to the present order of approximation is given by

$$u = 1 - \frac{A}{x} e^{-r^2/4x} + \frac{\ln x}{x^2} G(\bar{\zeta}) + \frac{1}{x^2} H(\bar{\zeta}) \quad (79)$$

where $\bar{\zeta} = \eta^2/2 = r^2/4x$ and $G(\bar{\zeta})$ and $H(\bar{\zeta})$ are given by Eqs. (71) and (78), respectively. Thus, we can also write the solution as

$$u = 1 - \frac{A}{x} e^{-\bar{\zeta}} - \frac{1}{4} A^2 (\bar{\zeta} - 1) e^{-\bar{\zeta}} \frac{\ln x}{x^2} + \frac{1}{x^2} H(\bar{\zeta}) \quad (80)$$

On the axis $\bar{\zeta} = r = 0$, this reduces to

$$u(x,0) = 1 - \frac{A}{x} + \frac{1}{4} A^2 \frac{\ln x}{x^2} + \frac{\beta}{x^2} \quad (81)$$

where β is an undetermined constant.

Let us now consider what happens in the next iteration, the determination of $\bar{u}^{(3)}$. Equation (48) now reads

$$\frac{\partial \bar{u}^{(3)}}{\partial x} - \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial \bar{u}^{(3)}}{\partial r} \right) = -\bar{u}^{(2)} \frac{\partial \bar{u}^{(2)}}{\partial x} - \bar{v}^{(2)} \frac{\partial \bar{u}^{(2)}}{\partial r} \quad (82)$$

For convenience in calculating the right-hand side, we introduce $\eta = r/\sqrt{2x}$ as an independent variable; this allows us to write

$$-\bar{u}^{(2)} \frac{\partial \bar{u}^{(2)}}{\partial x} - \bar{v}^{(2)} \frac{\partial \bar{u}^{(2)}}{\partial r} = -\bar{u}^{(2)} \frac{\partial \bar{u}^{(2)}}{\partial x} + \left(\frac{1}{2} \frac{\eta}{x} \bar{u}^{(2)} - \frac{\bar{v}^{(2)}}{\sqrt{2x}} \right) \frac{\partial \bar{u}^{(2)}}{\partial \eta} \quad (83)$$

In terms of η , Eq. (80) becomes

$$u = 1 - \frac{A}{x} e^{-\eta^2/2} - \frac{1}{4} A^2 \left(\frac{\eta^2}{2} - 1 \right) e^{-\eta^2/2} \frac{\ln x}{x^2} + \frac{1}{x^2} H(\eta) \quad (84)$$

It will be convenient for our purposes if we represent Eq. (84) by

$$u = 1 - \frac{D_1(\eta)}{x} - D_2(\eta) \frac{\ln x}{x^2} - \frac{D_3(\eta)}{x^2} = 1 + \bar{u} \quad (85)$$

Integrating the continuity equation, Eq. (42), and using Eq. (85), we find

$$v = \frac{-2}{\eta\sqrt{2x}} \left[\frac{1}{2x} D_1 \eta^2 + G_2(\eta) \frac{\ln x}{x^2} + \frac{G_3(\eta)}{x^2} \right] \quad (86)$$

where G_2 and G_3 are given in terms of D_2 and D_3 but their explicit representation is not necessary at this time.

If we now substitute Eqs. (85) and (86) into the right-hand side of Eq. (83), this equation becomes

$$\begin{aligned} -\frac{u^{(2)}}{u} \frac{\partial u^{(2)}}{\partial x} - \frac{v^{(2)}}{v} \frac{\partial u^{(2)}}{\partial r} = & \frac{1}{x^3} D_1^2 + \frac{\ln x}{x^4} \left[3D_1 D_2 - D_1' \left(\frac{G_2}{\eta} - \frac{1}{2} \eta D_2 \right) \right] \\ & + \frac{1}{x^4} \left[3D_1 D_3 + D_1 D_2 - D_1' \left(-\frac{1}{2} \eta D_3 + \frac{G_3}{\eta} \right) \right] + \dots \end{aligned} \quad (87)$$

The right-hand side of Eq. (87) is now substituted for the right-hand side of Eq. (82) and is therefore the inhomogeneous term for the third iteration. The term D_1^2/x^3 is exactly the one treated earlier in the second iteration (see Eq. (58)), and since $F(x,r)$ enters Eq. (57) (or Eq. (60)) linearly, it will contribute only the terms obtained in the second iteration. Hence, we shall limit our attention to the remaining two terms on the right-hand side of Eq. (87).

We now let

$$F(x,r) = g_1(x)F_3(\eta) + g_2(x)F_4(\eta) \quad (88)$$

where

$$g_1(x) = \frac{\ln x}{x^4} \quad g_2(x) = \frac{1}{x^4}$$

and F_3 and F_4 are the coefficients in Eq. (87), multiplying g_1 and g_2 , respectively. Correspondingly, we define

$$L_1'(x) = xg_1(x) = \frac{\ell n x}{3x}$$

$$L_2'(x) = xg_2(x) = \frac{1}{3x}$$

so for large x

$$\left. \begin{aligned} L_1(x) &\sim -\frac{1}{2} \frac{\ell n x}{x^2} - \frac{1}{4x^2} \\ L_2(x) &\sim -\frac{1}{2x^2} \end{aligned} \right\} \quad (89)$$

Now consider the second term of Expression (63), giving the contribution from $F(x,r)$ to u ; in particular, we take the sum of two such terms, $L_1(x)$ and $F_3(\eta)$ appearing in one and $L_2(x)$ and $F_4(\eta)$ in the other:

$$\begin{aligned} & - \int_0^x \frac{L_1(x')}{(x-x')^2} dx' \int_0^\infty d\eta' \left[1 - \frac{x\eta'^2}{2(x-x')} \right] \eta' F_3(\eta') e^{-\frac{x'\eta'^2}{2(x-x')}} \\ & - \int_0^x \frac{L_2(x')}{(x-x')^2} dx' \int_0^\infty d\eta' \left[1 - \frac{x\eta'^2}{2(x-x')} \right] \eta' F_4(\eta') e^{-\frac{x'\eta'^2}{2(x-x')}} \end{aligned}$$

For large x the leading terms are

$$\frac{1}{x^2} \left\{ \int_0^x L_1(x') dx' \int_0^\infty \left(1 - \frac{\eta'^2}{2} \right) \eta' F_3(\eta') d\eta' \right.$$

$$\begin{aligned}
 & + \frac{1}{x} \left[\int_0^x L_1(x') x' dx' \int_0^\infty \left(2 - 2\eta'^2 + \frac{\eta'^4}{4} \right) \eta' F_3(\eta') d\eta' \right] \\
 & + \int_0^x L_2(x') dx' \int_0^\infty \left(1 - \frac{\eta'^2}{2} \right) \eta' F_4(\eta') d\eta' \\
 & + \frac{1}{x} \left[\int_0^x L_2(x') x' dx' \int_0^\infty \left(2 - 2\eta'^2 + \frac{\eta'^4}{4} \right) \eta' F_4(\eta') d\eta' \right] \Bigg\} \quad (90)
 \end{aligned}$$

According to Eq. (89),

$$\int_0^x L_1(x') dx' \sim \frac{\ln x}{2x} + \frac{3}{4x}$$

$$\int_0^x L_2(x') dx' \sim \frac{1}{2x}$$

$$\int_0^x L_1(x') x' dx' \sim -\frac{1}{4} (\ln x)^2 - \frac{1}{4} \ln x$$

$$\int_0^x L_2(x') x' dx' \sim -\frac{1}{2} \ln x$$

Substitution of these integrals into expression (90) allows us to write it as

$$\begin{aligned}
 & - \frac{(\ln x)^2}{4x^3} \int_0^\infty \left(2 - 2\eta'^2 + \frac{\eta'^4}{4} \right) \eta' F_3(\eta') d\eta' + \frac{\ln x}{2x^3} \left[\int_0^\infty \left(1 - \frac{\eta'^2}{2} \right) \eta' F_3(\eta') d\eta' \right. \\
 & \left. - \frac{1}{2} \int_0^\infty \left(2 - 2\eta'^2 + \frac{\eta'^4}{4} \right) \eta' F_3(\eta') d\eta' - \int_0^\infty \left(2 - 2\eta'^2 + \frac{\eta'^4}{4} \right) \eta' F_4(\eta') d\eta' \right] \\
 & + \frac{1}{2x^3} \left[\frac{3}{2} \int_0^\infty \left(1 - \frac{\eta'^2}{2} \right) \eta' F_3(\eta') d\eta' + \int_0^\infty \left(1 - \frac{\eta'^2}{2} \right) \eta' F_4(\eta') d\eta' \right] \quad (91)
 \end{aligned}$$

Thus we see that in the solution for $\bar{u}^{(3)}$, terms of order $(\ln x)^2/x^3$, $\ln x/x^3$, and $1/x^3$ appear.

To gain some insight into why terms involving powers of the natural logarithm should appear in the expansion, let us return to Eq. (48). The complementary solution at any stage of the iteration is determined as the solution of Eq. (48) with the right-hand side set equal to zero. In terms of the variables x and $\eta = r/\sqrt{2x}$, the complementary function satisfies the equation

$$\left(\frac{\partial}{\partial x} - \frac{1}{2} \frac{\eta}{x} \frac{\partial}{\partial \eta} \right) \bar{u}^{(n)} - \frac{1}{2x\eta} \frac{\partial}{\partial \eta} \left(\eta \frac{\partial \bar{u}^{(n)}}{\partial \eta} \right) = 0 \quad (92)$$

If we now substitute

$$\bar{u}^{(n)} = \frac{Q_n(\eta)}{x^n} \quad (93)$$

we obtain the following equation for $Q_n(\eta)$:

$$\eta Q_n'' + (\eta^2 + 1) Q_n' + 2n\eta Q_n = 0 \quad (94)$$

Now let

$$Q_n(\eta) = M_n(\eta) e^{-\eta^2/2} \quad (95)$$

This yields

$$M_n'' + \left(\frac{1}{\eta} - \eta\right) M_n' + 2(n-1)M_n = 0$$

Finally, we again introduce the independent variable $\zeta = \eta^2/2$ in terms of which the equation for M_n becomes

$$M_n''(\zeta) + (1 - \zeta)M_n'(\zeta) + (n-1)M_n(\zeta) = 0 \quad (96)$$

The solution of this is

$$M_n(\zeta) = L_{n-1}(\zeta)$$

where $L_n(\zeta)$ is the Laguerre polynomial of order n . Hence

$$Q_n(\eta) = L_{n-1}\left(\frac{\eta^2}{2}\right) e^{-\eta^2/2} \quad (97)$$

The boundary conditions to be satisfied by $Q_n(\eta)$ are

$$Q_n'(0) = 0 \quad Q_n(\eta) \rightarrow 0 \text{ exponentially as } \eta \rightarrow \infty$$

The second boundary condition is immediately seen to be satisfied. For the first condition, we note that

$$Q_n'(\eta) = \eta L_{n-1}'\left(\frac{\eta^2}{2}\right) e^{-\eta^2/2} - \eta L_{n-1}\left(\frac{\eta^2}{2}\right) e^{-\eta^2/2}$$

so $Q_n'(0) = 0$, and this condition is also satisfied for any n . Hence, the complementary functions for all n exactly satisfy the boundary conditions at $\eta = 0$ and $\eta = \infty$.

The corresponding inhomogeneous equation looks like

$$\eta R_n'' + (\eta^2 + 1)R_n' + 2n\eta R_n = F(\eta) \quad (98)$$

where $F(\eta)$ is a known function of η , perhaps involving arbitrary constants. If we let

$$R_n(\eta) = S_n(\eta)e^{-\eta^2/2} \quad (99)$$

and also introduce the variable $\zeta = \eta^2/2$ again, the equation becomes

$$\zeta S_n''(\zeta) + (1 - \zeta)S_n'(\zeta) + (n - 1)S_n(\zeta) = \frac{F(\zeta)e^\zeta}{2\sqrt{2\zeta}} \quad (100)$$

In particular, we consider the third approximation, $n = 3$. The equation is then

$$\zeta S_3'' + (1 - \zeta)S_3' + 2S_3 = \frac{F(\zeta)e^\zeta}{2\sqrt{2\zeta}} \quad (101)$$

If we set

$$S_3(\zeta) = L_2(\zeta)T(\zeta) = (\zeta^2 - 4\zeta + 2)T(\zeta) \quad (102)$$

($L_2(\zeta)$ is the Laguerre polynomial of order 2), then we obtain

$$T'' + \left(\frac{-\zeta^3 + 9\zeta^2 - 14\zeta + 2}{\zeta(\zeta^2 - 4\zeta + 2)} \right) T' = \frac{F(\zeta)e^\zeta}{2\zeta\sqrt{2\zeta}(\zeta^2 - 4\zeta + 2)}$$

The first integral of this equation is given by

$$e^{-\zeta}\zeta(\zeta^2 - 4\zeta + 2)^2 T' = \int_0^\zeta \frac{F(\zeta)(\zeta^2 - 4\zeta + 2)}{2\sqrt{2\zeta}} d\zeta \quad (103)$$

Thus, if R_3 is to tend to zero exponentially as ζ or $\eta \rightarrow \infty$, we must require

$$\int_0^{\infty} \frac{F(\zeta)(\zeta^2 - 4\zeta + 2)}{2\sqrt{2}\zeta} d\zeta = 0 \quad (104)$$

If $R_3(\eta)$ represents the coefficient of the term $\ln x/x^3$ in $\bar{u}^{(3)}$, that is,

$$\bar{u}^{(3)} = \frac{\ln x}{x^3} R_3(\eta) + \text{term of } O(x^{-3})$$

then the $F(\zeta)$ to be entered in Eq. (98) and the following equations is the coefficient of $\ln x/x^4$ in Eq. (87). The explicit representation of this in terms of η is

$$3D_1D_2 - D_1' \left(\frac{G_2}{\eta} - \frac{1}{2} \eta D_2 \right) = A^3 \left(\frac{1}{2} \eta^2 - 1 \right) e^{-\eta^2}$$

Hence

$$F(\zeta) = A^3(\zeta - 1)e^{-2\zeta} \quad (105)$$

If we substitute this expression into the integral in Eq. (104), the integral becomes

$$\frac{A^3}{2\sqrt{2}} \int_0^{\infty} \frac{(\zeta - 1)(\zeta^2 - 4\zeta + 2)e^{-2\zeta}}{\sqrt{\zeta}} d\zeta$$

The value of this integral is $(-89/216)A^3$, and hence the condition that $R_3 \rightarrow 0$ exponentially as $\eta \rightarrow \infty$ is not satisfied.

This now explains the appearance of the $(\ln x)^2/x^3$ term in the solution for $\bar{u}^{(3)}$. This term must be introduced to insure that the particular integral corresponding to $\ln x/x^3$ be exponentially small when η is large. This $(\ln x)^2/x^3$ term consists only of the complementary function (solution of Eq. (94)), its numerical factor being determined from the condition on the particular integral corresponding

to $\ln x/x^3$. The factor multiplying the complementary function corresponding to $\ln x/x^3$ is determined by the condition on the particular integral corresponding to $1/x^3$. However, there is no way of obtaining the numerical factor in the complementary function corresponding to $1/x^3$ in this way, and as we have already seen, it probably depends on the initial profile, $f(r)$.

Generally, difficulty arises each time a complementary solution to Eq. (48) of the form $x^{-n}G_n(\eta)$ satisfies the conditions $G_n'(0) = 0$ and $G_n(\eta) \rightarrow 0$ exponentially as $\eta \rightarrow \infty$. We have already shown that this occurs for all integers n , and thus at each step of the solution it may be necessary to introduce an extra factor $\ln x$ into the series. At the same time, the x^{-n} term will contain an indeterminate numerical factor depending in some way on the initial profile of u .

IV. DISCUSSION

A number of differences may be pointed out between the two-dimensional and axisymmetric results. Stewartson⁽⁵⁾ shows that the velocity at $y = 0$ for the flat-plate asymptotic far-wake solution is

$$u(x,0) = 1 - \frac{\alpha}{x^{1/2}} - \frac{\alpha^2}{2x} - \frac{\alpha^3}{8\sqrt{3}} \frac{\ln x}{x^{3/2}} + \frac{\beta}{x^{3/2}} + O\left(\frac{\ln x}{x^2}\right)$$

where α is a known constant related to the drag of the plate and β is an unknown constant. (Stewartson gives a coefficient of the term $\ln x/x^{3/2}$ which is twice as great as the one given above; this was due to a slight error in one of his equations that was corrected by Crane.⁽⁹⁾ We have found the corresponding axisymmetric solution to be Eq. (81):

$$u(x,0) = 1 - \frac{A}{x} + \frac{1}{4} A^2 \frac{\ln x}{x^2} + \frac{B}{x^2} + O\left(\frac{(\ln x)^2}{x^3}\right)$$

where A is known in terms of the drag of the body and B is unknown.

We note that the indeterminacy in the asymptotic expansion appears earlier in the axisymmetric than in the two-dimensional case, occurring in the fourth term in the former and in the fifth in the latter. Also, in the two-dimensional case, eigenfunctions occur for terms involving $x^{-n/2}$ only when n is an odd integer, so a factor $\ln x$ may be introduced at alternate steps of the series expansion. In the axisymmetric case, on the other hand, eigenfunctions exist for each n , so these additional logarithmic factors may occur at each step.

As indicated in Section II, the pressure in the wake is constant to order x^{-2} ; consequently, Eqs. (80) and (81) up to terms of order $x^{-2} \ln x$ represent the asymptotic far-wake solution at large Reynolds number for any finite axisymmetric body. The term of order x^{-2} , which is the first term in the asymptotic expansion affected by a nonconstant pressure, is already indeterminate due to the existence of an eigen-solution; thus, the determinate part of the solution is unaffected by the existence of a nonzero pressure gradient.

It is of some interest to compare the approach taken here with that of Chang⁽⁶⁾ and Childress⁽⁷⁾ in their analyses of the far flow field for arbitrary Reynolds number. As indicated earlier, these authors introduce an artificial parameter ϵ , which is the ratio of a characteristic length to an artificial length, and recast the problem of the flow at large distances as a perturbation for small values of ϵ ; in other words, the coordinate-type expansion is replaced by a parameter-type expansion. They then construct two expansions for the velocity and two for the pressure, an outer expansion valid outside the wake region and an inner expansion valid within the wake region. The domain of validity of these two expansions overlaps, and hence an expansion uniformly valid at large distances may be constructed by a combination of outer and inner expansions.

Apart from the usual principles for matching inner and outer expansions, Chang and Childress employ two additional, independent principles for finding the form of the expansions and for eliminating certain apparent indeterminacies. These are (1) the principle of eliminability, and (2) the principle of rapid (transcendental) decay of vorticity. The first of these requires that the artificial parameter ϵ be eliminable from the expansions. The second principle, that for a finite or semi-infinite solid in a uniform stream the vorticity decays at an exponential rate with distance outside the wake (or at least faster than any power of the distance), is assumed to hold for solutions of the Navier-Stokes equations; all the known exact and linearized solutions support this assumption (see Ref. 6). The application of the eliminability principle leads to the concept of "switchback terms" which must be introduced in order that ϵ be eliminable from the resulting expansions. In the Chang and Childress analyses, the effect of switchback is to introduce logarithmic terms; thus, for example, in the axisymmetric solution of Childress, terms $\epsilon^i (\ln \epsilon)^j$ (i and j are integers satisfying $j \leq i - 1$) are introduced into expansions which originally contained only integral powers of ϵ . Childress does not attempt to explain precisely the reason for switchback but notes that it is a nonlinear phenomenon.

One of the essential differences between the artificial-parameter analyses of Chang and Childress and the present analysis, based on a coordinate-type perturbation, is the reason for the introduction of the logarithmic terms. In the present analysis these terms are introduced in order that the vorticity decay exponentially at infinity. In the artificial-parameter analysis, as indicated above, the switchback terms arise from the application of the principle of eliminability of the artificial parameter; the principle of rapid decay of vorticity is put forth as a completely independent principle. Thus, the underlying reason for the occurrence of the switchback terms, which is not fully explained in the Chang and Childress analyses, would seem to stem from only one basic principle--rapid decay of vorticity.

It might be added that after listing certain of the advantages of the artificial-parameter approach, Chang indicates that a direct coordinate-type procedure would seem preferable for a final analysis.

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